Writing Math Research Papers: A Guide for High School Students and Instructors Fifth Edition

This is dedicated to the ones I love—
Linda, Julie, Michael, Jordan, and Courtney

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INTRODUCTION

As an avid fan of writing and mathematics, I began giving my students writing assignments as a new teacher in 1977. The initial reaction was "This is *math*— why are we doing English?" As the students completed their math writing projects on varied topics such as automobile insurance, income taxes, stocks and bonds, consumer credit, and more, they began to realize the value of being able to communicate the powerful results mathematics can provide. Students who wrote about surface area, parabolas, the Pythagorean theorem, and other mathematics topics were quick to point out that they internalized the material so much better because in order to explain it they *really* had to understand it. Students concluded that writing is an essential component of any discipline, not just humanities courses. Based on the successes of these early writing experiences, I began to incorporate writing in my mathematics classes on a daily basis. Needless to say, I was very excited when the National Council of Teachers of Mathematics (NCTM), an organization of educators dedicated to the improvement of mathematics instruction, recommended the integration of writing across the mathematics curriculum. Mathematics and writing, to me, have always complemented each other naturally.

Since problem solving is a central focus of math classes, it is clear that writing must be integrated with mathematics courses. Problem solvers have to be able to explain the procedures they used and the solutions they found.

Have you ever been absent from a class and asked a fellow student, "What did we do in math yesterday?" only to hear the response "I know it, but I can't explain it"? How did you react? Imagine a detective telling a superior, "I've solved the case, but I can't explain my solution"!

Mathematicians must be able to communicate their findings to others so their important results can be used to solve future problems. This is not the only value of writing in mathematics. Writing will also help *you* understand mathematical concepts. As you construct written explanations, you will need to explore and review mathematical concepts in your mind. *Writing Math Research Papers* will help you expand your ability to read about mathematics, explore mathematics, and write about your mathematical

Writing Math Research Papers, pages ix–xix.

experiences. Writing in mathematics also aligns beautifully with the recently implemented Common Core State Standards for Mathematical Practice.

MATH RESEARCH AND THE COMMON CORE STANDARDS FOR MATHEMATICAL PRACTICE

The Common Core State Standards were introduced in 2010 and subsequently adopted by over 40 states. These Standards require that these eight principles of mathematical practice (MP1–MP8) become an integral part of all mathematics curricula:

MP1. Make sense of problems and persevere in solving them.
MP2. Reason abstractly and quantitatively.
MP3. Construct viable arguments and critique the reasoning of others.
MP4. Model with mathematics.
MP5. Use appropriate tools strategically.
MP6. Attend to precision.
MP7. Look for and make use of structure.
MP8. Look for and express regularity in repeated reasoning.

Not all mathematics curricula adequately addressed these requirements. Consequently, thousands of schools all over the United States revised their curricula to incorporate the Common Core recommendations. The Math Research curriculum, as presented in *Writing Math Research Papers*, aligns so perfectly with the eight standards that nothing in the curriculum needed to be "adjusted" to meet MP1- MP8. Math Research is a natural forum for these standards, because it takes an authentic look at how mathematical findings are developed, proven, and extended.

WHY DO MATH RESEARCH?

If you've been successful in mathematics and you enjoy it, you may wonder: "What other mathematical challenges can I explore? How can I tap my enthusiasm in mathematics to further my education?" Math research will provide you with the opportunity to explore mathematics and enjoy the thrill of discovery. Your exploration of a new concept will give you experience in tackling any non-routine problem you may encounter. Throughout this book, you will be introduced to the world of research. By learning problem-solving and research skills, you'll vastly increase your potential to solve all types of problems.

You already possess many of the tools necessary to do exploratory mathematics. Perhaps you have done some problem solving in your math class. Your math research paper will be a major project—you will not finish it in one day or even one week. It will require you to read, write, think, and investigate mathematical ideas. It will improve your writing, reading, and oral communication in other subjects as well. It will empower you to create, conjecture, challenge, and question according to your ability,

motivation, priorities, and schedule. Research will expose you to the beauty and practicality of mathematics and reward you with the tremendous feeling of uncovering a result previously unknown to you. Communicating your experiences through the writing of the research paper will help you understand and appreciate the mathematics you've explored as well as help others learn about your findings.

Your instructor can serve as coach, and this book can serve as a guide, but the driving force behind your research is your own motivation. You will become an expert on your topic because you will spend a great deal of time exploring it. You will learn mathematics by *doing* mathematics.

WHAT'S IN THIS BOOK?

Often, when mathematics "term papers" are assigned, students are given a list of topics and a due date. Specific instruction on reading, extending, and writing mathematics may not be offered. Writing mathematics and doing mathematics research involve sophisticated skills that are not innate. The purpose of *Writing Math Research Papers* is to introduce you to these skills and give you logical, sequential direction in developing valuable skills that will last a lifetime. We examine the purpose of each chapter here to better acquaint you with how these skills will be developed.

Chapter 1: Mathematics: Shouting Questions and Whispering Answers

As the title implies, as you engage in mathematical thinking it is natural for many questions to arise. Many of these questions are open-ended, challenging inquisitions. No matter how deeply you delve into a topic, new questions will always surface; hence, mathematics *shouts* questions. Finding the answers to these questions requires time, effort, and skill. The answers do not surface as quickly as the questions; hence, mathematics *whispers* answers. The chapter contains some examples that show that even basic arithmetic concepts can inspire intriguing questions.

Chapter 2: Finding a Topic

Many students are first exposed to research in history and science courses. Some students have even written research papers and conducted research experiments in these fields. The difference between a report and a research paper is discussed in this chapter. Often, students who write math papers pick a topic that is too broad. As this chapter points out, topics for research papers must be specific. There are many sources for succinct topics for research papers. Mathematics journals written for instructors and math enthusiasts feature short articles that describe particular mathematical concepts. These articles, usually three to six pages long, can provide an excellent springboard for your paper. Chapter 12 lists journals you can use as sources for your paper's topic. Other suggestions for finding topics are also discussed in this chapter. In summary, your re-

search paper will build upon a short article, problem, or idea, rather than attempt to condense information from several immense library books.

Chapter 3: Problem Solving: A Prerequisite for Research

Problem solving has always played an integral part in mathematics courses. You will encounter non-routine problems on the job and in everyday life, and the solutions to these problems will be crucial. Solid problem-solving experience is advantageous. This chapter reviews problem-solving strategies you may have learned in some of your previous math courses. Familiarity with these strategies will help you answer some of the questions that arise as you carry out your research. Additionally, you will use these problem-solving strategies throughout your entire life!

Chapter 4: The Importance of Clear Communication

This chapter underscores the need to express your thoughts coherently and comprehensively. Some of the mathematics you are researching may be pretty difficult to understand, so it needs to be explained clearly to the reader. Communication is an art and a skill, just like playing the piano, pitching, or plumbing, and it can be learned, practiced, critiqued, improved, and mastered.

Chapter 5: The Notes You Take in Math Class

This chapter provides several activities for writing about mathematics that you can use to improve your writing even if you don't intend to write a math research paper right away. They will improve your note-taking in class as well as help you communicate your thoughts clearly whenever you write about mathematics. These writing activities concentrate on mathematics you are already familiar with from your core math class, so you can concentrate solely on improving your writing skills.

Chapter 6: Technical Writing Techniques

This chapter gives specific writing strategies that can be used to achieve the characteristic formatting you saw throughout the projects in Chapter 5. Technical writing has its own set of rules, which makes it slightly different from other forms of writing you have learned in school, including creative writing, poetry, document-based essays, etc.

Chapter 7: Conjectures, Theorems, and Proofs

The similarities and differences between history, science, and mathematics research and the role of proofs in each are discussed in this chapter. In your mathematics reading, you will encounter proofs. You should try to read through them and understand them so you can explain them in your paper. You may come up with some questions of your own and even make conjectures (hypotheses) about the answers to these questions. As you become more experienced, you may actually prove your conjectures with

original proofs. This chapter will expose you to the role that proofs can play in your research. Your experience with proof will be invaluable to you in future math courses.

Chapter 8: Reading and Keeping a Research Journal

You might not understand how to turn a short article into a research paper, but as you read through this chapter you will see that you might be able to read fifty pages of a novel in less time than it takes to carefully read and digest one page or even one paragraph of mathematics. Specific tips for reading mathematics journal articles are given. Because you will not start writing the formal paper until part of your research is completed, you will need a comprehensive set of notes on your research. Thus, you will keep a journal as you read. You will keep anything your write about, including scrap notes and errant trials.

Chapter 9: Components of Your Research Paper

Chapters 4, 5, and 6 introduce you to writing mathematics, and Chapters 6, 7, and 8 instruct you in how to conduct your research in a logical fashion. Chapter 9 helps you pull it all together for the formal paper. The parts of the research paper are discussed. As you read this chapter, you will also refer to Chapter 11, which features samples of actual student work that will help you get a feel for how the formal paper should appear.

Chapter 10: Oral Presentations

Many students are given a chance to present their research at special events and mathematics competitions. Giving an oral presentation requires extensive planning, visual aids, and practice. This chapter will give you tips in staging a polished, professional-quality presentation that reflects the high quality of your research.

Chapter 11: Sample Pages from Student Research Papers

Chapter 11 shows some actual pages from math research papers done by high school students. These samples will help you get used to the formatting, language style, tables, diagrams, and other idiosyncrasies inherent in technical writing. You can use them as references when you write.

Chapter 12: Some Suggested Resources

This chapter offers suggestions for specific journals, books, periodicals, and websites that can be used to find topics, enhance papers, extend papers, and run your research course. Recommendations and citations for specific journal articles appropriate for high school students are also included.

Chapter 13: A Guide for Instructors and Administrators

Creating and implementing a new program such as a math research course is a major undertaking. Preparing scheduling, logistics, enrollment, technological necessities, and physical space require careful planning. Chapter 13 includes suggestions for instituting your math research program.

After reading the purpose of each chapter, you may wonder, "Will doing a math research paper strengthen my mathematics education? If so, how?"

IMPROVING YOUR MATHEMATICS EDUCATION IN MANY FACETS

The skills you will acquire during your research project are considered very valuable by mathematics educators and can be applied to other mathematics courses as well as to other disciplines. Your research project reflects an effective, practical incorporation of many mathematics education principles in a format that will allow you to experience some of the power of the various teaching and learning strategies used in mathematics. Your research topic may be a mathematical application to another discipline or a pure mathematics topic—one that advances knowledge about a certain mathematics concept rather than solving a practical problem in another discipline. In either case, the research strategies you learn will be valuable in virtually all mathematical situations you encounter. Chapter 2: Finding a Topic will help you decide what road you should take. You will be traveling down a mathematical highway illuminated by communication, reasoning, connections, problem solving, representation and modeling.

Communication and Your Research

Researchers need to be communications-minded. They must read, attend lectures, listen, write clearly, and make oral presentations of their work. Researchers need to digest information, process it, perform their work, and report their findings. You will benefit from others' ability to communicate because, early in your research, you will be reading about your topic. Chapter 9: Reading and Keeping a Research Journal examines specific reading skills you can adopt to help in reading technical writing. As you read, you will ask yourself informal questions to help you understand passages as well as formal questions as mathematical extensions of your readings. As you explain to others where your questions lead you, your explanations must be clear to outside readers. Good communication skills are essential to effective presentations, both written and oral. Chapters 4, 5, 6, 10, and 11 delve into the communication arts with respect to your research. Above all, since clarity is paramount, your organization, definitions, questions, proofs, conjectures, and explanations must be well written and logically organized. You'll need to reflect on each stage of your work carefully before you can put it into your own words. You'll need to use notation, graphs, and other forms of representation effectively in order to convey mathematics succinctly. Becoming adept

at communicating your work will allow others to benefit from the knowledge you've acquired. Other student-researchers may want to continue your work, extend it, alter it, or make new investigations. They will rely on your communication skills.

Reasoning and Your Research

Students need to develop an ability to make conjectures, interpret claims, justify claims, and communicate their findings. As you read through your research articles, you will need to follow the arguments presented by the authors. You must make sure that their arguments are valid and that you understand the logic used by testing their claims and working through their proofs step-by-step. You will formulate ideas based on patterns, your mathematical intuition, and the mathematical tools you have acquired in your coursework. Making conjectures requires reasoning—conjectures aren't guesses but rather hypotheses that, whether true or false, can reasonably be tested. Mathematicians make many conjectures that turn out to be false. If, after working with a conjecture, you suspect that it is false, you might try to find a counterexample—a single case in which the conjecture is not true—or explain theoretically why your original suspicions were not true in every case. If you are convinced that your conjecture is true, you may try to construct a proof—a valid argument that your hypothesis is indeed a theorem. There are different types of proofs for different conjectures. You might read a statement that is not proved and decide to construct a proof on your own. This proof then becomes part of your research. Chapter 6 gives an overview of the use of proofs in mathematics.

Connections and Your Research

As you explore your paper's topic, you will have the chance to integrate different branches of mathematics in your research. Students doing research on a geometry topic might use algebra, logic, calculus, trigonometry, set theory, and more to create proofs and give explanations about their topic. Tapping the different fields as they are needed requires knowledge and discretion, as well as a well-equipped mathematical tool kit. In this respect, your paper is different from the study of a single unit in a math course. You may need to learn part of a topic on your own (with the assistance of your instructor and an appropriate textbook) because it can help your research. You may be able to test or prove one of your conjectures in two different ways—for example, using coordinate geometry and plane geometry. As part of your research, you might discuss which method was easier, better, faster, shorter, more intuitive, and so on.

If your paper deals with a mathematics application to another discipline such as psychology, business, or science, you will be making connections not only within mathematics but between mathematics and the discipline you're modeling. When researching an applied-math topic, you need to become knowledgeable about the discipline you are researching as well as the mathematics you are using. The connections are seemingly endless:

- What must an architect know about an ellipse in order to design a whispering gallery?
- What is the shape of the suspended cables of a suspension bridge?
- How are seismographs, rates, and circles used to find the epicenter of an earthquake?
- How can mathematics be used to find the area of an irregular shape such as a golf green?
- How are graphs and statistics used to predict economic trends?
- How can doctors use conic sections to break up kidney stones without invasive surgery?
- How are paraboloids used to create telescopes that can take pictures deep into space?

As society becomes more technologically oriented, mathematics assumes a more prevalent role in the progress of other disciplines. A connection to mathematics is an essential component of the research that will advance knowledge in other fields. Use your communication skills to help your readers make connections *in* your work and *to* your work.

Problem Solving and Your Research

Your readings will include passages that you don't immediately understand. As you reread certain sentences several times, you will need to employ your problem-solving skills to figure out their meaning. The passage you are having trouble with in effect becomes its own problem. Attack it with determination. Your readings will include many such hurdles. You might even create some hurdles yourself, since each concept you understand and internalize may breed more questions and possible extensions. Such questions and extensions are really new problems. Pose them to your readers to investigate on their own, or raise their solutions and address their solutions in your research. Chapter 3 provides you with an overview of problem-solving strategies.

Representation, Modeling, and Your Research

You are already familiar with mathematical representation. If you saw a foreign mathematics textbook, you'd realize how adept you are at reading mathematical thoughts. Each year, I expose my students to a few pages of mathematics textbooks written in other languages, even languages that use other types of letters. The students are always amazed at how much they can understand in a French, Hebrew, or Chinese textbook. How is this possible?

Mathematics is full of representations. All the symbols and diagrams we use represent something, and these representations are often universal. A representation could have some interesting historical origin, or it could simply be a convenient shorthand used to represent some fact. (Think of how the percent symbol, %, actually incorporates the 1

and two 0s from the number 100.) A model is a representation. Some models are diagrams—physical representations—for instance, a perspective drawing of a cube. Other models use symbols. The equation $C = 3v + 10$ could be used to model the cost C of downloading v videos from a movie club each year, if the movie club has a $10 annual membership fee, and they charge $3 for each download. In statistics, you can represent an average as a mean or a median, and you must interpret the situation to apply the best representation. You can oversimplify a problem or actually do it incorrectly by using an incorrect representation. Representations are the tools that expand your capacity to think mathematically. They are the numbers and the concepts behind them; they are algebraic symbols, expressions, equations, and graphs; they are geometric terms and models; they are statistical formulas and displays. Representation is essential to understanding and communicating mathematics and to applying mathematics to real-life situations. To describe a situation mathematically, you need to represent the situation using the language of mathematics.

Your research in mathematics may include investigating notation new to you. You may also invent new notation to help you organize your findings as you answer a research question. You will use representations to communicate your research results. As you research and present your findings, you will be choosing among different forms of representation. You can use graphs, pictures, diagrams, lists, and narratives. With new technology, including graphing calculators and dynamic geometry software, you might find that your investigation leads you to new notations and terminologies. Think of all the words in your daily technical vocabulary that most people didn't know twenty years ago—hashtag, Twitter, smart phone, download, instagram, tweet, and so on. Think of what "closing a window" meant thirty years ago, and how the same phrase today connotes working on a computer! New notations, processes, and terms will lead you to form new questions.

Chapters 3–6 will require you to examine the use of mathematical representation as you engage in your research, and the culmination of your research—your oral presentation—will depend on good representation to be clear and accurate.

HOW TO USE *WRITING MATH RESEARCH PAPERS*

Ideally, you will use *Writing Math Research Papers* in a group setting, with your classmates and instructor giving you feedback and coaching. If you are planning to write a math research paper, you and your classmates should read *Writing Math Research*

Papers gradually as you go through the research process. The following time line is offered as a general guide to your project. Let's look at how a nine-month school year could be scheduled.

Month	Activity
1, 2	Read and discuss Writing Math Research Papers, Chapters 1–7, and 12. Find a topic and an article. Make copies of your article, and start a Math Author Project using a topic from your core mathematics class. Start your bibliography. These activities do not have to be finished within the first two months—they need to be started.
2, 3, 4, 5	Begin reading your article, taking notes, testing claims, and looking for patterns. Keep a journal of all your writings. Save your annotations of the article. Begin periodic (usually weekly) consultations. Read Chapter 9.
4, 5, 6, 7	Continue reading your article, writing up findings, keeping your journal, and consultations. Write your Problem Statement. Compile all of your findings to date for your Related Research section.
7, 8, 9	Read Chapter 10 and start planning your oral presentation. Present paper to an audience and record the presentation. Watch it and critique it.

As you progress through each stage of your research, you should reread previous chapters and use them as a reference. It is based on a two-semester course, but can be adjusted to meet your specific situation. The time line can help you prorate your time if you are not writing your paper over a full two-semester period. The amount of time you have, the amount of coaching you receive, and the topic you choose will all affect the amount of time you spend on any one activity. Always remember that research projects require flexibility. You can never "guarantee" when you are going to come up with certain findings or proofs. However, you *can* guarantee the time, trials, and effort you put in. The following are some suggestions for other ways you could use the book:

- If you are not sure whether you are going to write a research paper, reading through Chapters 1, 2, 8, and 9 can help you decide if you should undertake such a project.
- If you want to improve your mathematics writing skills by practicing writing about the mathematics you have already learned, read about the writing mathematics activities in Chapters 4, 5 and 6.
- If you would like to improve your ability to take notes in mathematics classes, read and do the activities in Chapters 4, 5 and 6.
- If you are an instructor or administrator planning a course in problem solving and/or mathematics research, read the entire book, with an emphasis on Chapter 13: A Guide for Instructors and Administrators.
- If you are an instructor coaching a student who is doing an independent-study math research paper, Chapters 2 and 12 will help you and your student find a

topic. Chapter 13 will help you understand the role you can play as this student's mentor.

- If you are a pre-service teacher in a college mathematics education program, you should read the entire book to familiarize yourself with what your future students may be doing. Perhaps you never wrote a math research paper, and most of the process is new to you, too. You may want to try writing your own research paper following the steps outlined in the book. Keep in mind that Chapters 4, 5 and 6 have excellent tips you can employ to help all of your students take better notes. Chapters 12 and 13 will help you with resources and classroom logistics. As you enter the world of mathematics writing and mathematics research, keep in mind that fellow educators and students are always interested in your ideas, successes, and suggestions.

"What we have to learn to do, we learn by doing."

—*Aristotle*